Shock Waves Ahead of a Fan with Nonuniform Blades

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When a fan operates at supersonic tip speeds, shock waves are generated ahead of the blades. If these blades are nonuniform, then the shock waves are also nonuniform, and tones at harmonics of the fan rotational frequency are generated. This paper presents a simple theory for the relation between the strengths of the individual shock waves, the blade stagger angles and the blade thicknesses, when the shock wave is detached from the blade leading edge. The analysis is in two parts: first a relation is found between the shock wave detachment and the blading nonuniformity. Then, weak shock wave theory is used to relate the detachment to the shock strength ahead of the fan. The results of the theory are in good agreement with experiments and so provide a theoretical basis for the blade shuffling procedures used to minimize blade-to-blade variations and to control shaft order tone generation by these variations.

I. Introduction

THIS paper discusses the generation of "buzz-saw" or "combination-tone" noise by supersonic compressors with nonuniform blading. If such a compressor has uniform blading, any tones generated are, by symmetry, at multiples of blade passing frequency only. The low harmonics of rotational frequency, that do in practice exist, are an important component of the aft cabin noise on the Lockheed L1011 TriStar. The causes of this buzz-saw noise in modern turbofan engines are the shock waves which appear ahead of the fan. These shock waves are generated with different amplitudes, depending on the blade parameters, and they then propagate nonlinearly along the fan inlet duct. It used to be thought that the low engine order tones were generated as a result of waveform distortion resulting from this nonlinear propagation, but recent work by Stratford and Newby[1] has shown that while the strengths of the shock waves change during this propagation process, the levels of the low-order harmonics of rotational frequency do not. The levels of these harmonics are, therefore, set by their levels at the fan disk. Stratford and Newby found a linear relation between shock amplitude and the differences in the stagger of adjacent blades. This is used[2] as the basis for controlling shock strengths and, hence, the low order harmonics upstream of the fan. The order of the fan blades is changed to minimize the second and third harmonics of the stagger angle pattern.

While this has been broadly successful in reducing the sound level in these harmonics, an additional factor is thought to be significant. This is the thickness of the blades. Recently, much work has been done to correlate the shock strengths with blade-to-blade thickness variations.[2] As a result, when the blades are shuffled, thickness as well as stagger angle variations are taken into account. In the Stratford and Newby paper, the experimental relation between blade angle and shock strength was at variance with existing theories. In these (see, e.g., Kurosaka[3]) the shock waves were assumed to be attached to the fan blades, so that if we neglect the losses through the shocks, the Mach number at a point is related directly to the local flow angle. The flowfield may then be solved by the method of characteristics. There, the Mach number and flow angle are constant along "characteristic" lines, at an angle $\theta = \cos^{-1}(1/M)$ to the flow. Using this property, and the conservation of mass flow, energy, and momentum across the shock waves, the complete flowfield ahead of the fan can be constructed. In practice, it is much easier to analyze the flow beyond the first blade spacing ahead of the fan using one-dimensional nonlinear acoustic theory (see Hawkings[4]).

Since the blades are only slightly cambered, they may be taken as flat, to a first approximation, and with one blade twisted from its original position the flow pattern close to the fan is as depicted in Fig. 1. Over the section AB, the flow ahead of the shock attached to the nth blade is uniform; all the changes in the flow behind it are caused by the expansion fan shed from its leading edge, where the flow is turned. Hence, the shock strength only depends on the blade angles and is, therefore, easily calculated. Since the shock waves are attached, they are uninfluenced by the flow downstream of the fan. Consider now the effect of changing the incidence of one blade; if the stagger angle of the nth blade is increased the Mach number ahead of the $(n+1)$th blade is reduced, and the Mach number behind the nth blade reduced. Consequently, the strength of the shock from the nth blade is increased and that from the $(n+1)$th blade reduced. Linearizing, we may then expect a relation of the form $\Delta P_n = A\alpha_n - B\alpha_{n-1}$ where ΔP_n is the shock amplitude, α_n the increase in stagger angle of the nth blade, and A and B are positive constants of the same order of magnitude.

This is in contradiction to the observed relation[1] $\Delta P_n = -0.194\,(\alpha_n - \alpha_{n-1})$, for which the slope has the opposite sign. Stratford and Newby argued that the difference arises because, in practice, the shock waves are detached from the blade leading edges. When a uniform fan or cascade is running, there is only one value of flow direction (for a given Mach number) for which the shock waves are attached—the so-called "unique incidence condition" (see, e.g., Ref. 5). In reality fans are operated way from this condition, at higher values of blade incidence. As Stratford and Newby demonstrate, this means that some of the inlet flow to a blade passage is spilled around the leading edge of the blade. This spillage is necessary to match the mass flows through the passages with those far upstream of the fan and it makes the blade behave, in effect, as a thick blunt-nosed body which has a detached shock ahead of it. A further consequence of shock wave detachment is that the shock waves are affected by the flow downstream of the fan. Stratford and Newby assert that this is the primary factor controlling the shock amplitudes. As the blade angles are altered, the change in swallowing capacity of a blade passage is proportional to the change in angle. This change in swallowing capacity then changes the spillage around the blade and alters the shock strengths accordingly. It is, therefore, plausibly responsible for the observed relation between shock strength and blade angles.

The aim of this paper is to model this phenomenon in a relatively simple way, and to justify quantitatively the observed experimental relations between shock strength, blade

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angles, and thicknesses. To do this, we assume that the blades are essentially thin and uncambered, and we consider only two-dimensional flow, such as persists near the tips of the blades where the experimental measurements are made. Further, we consider only the flow close to the fan since its upstream development is easily modeled by existing nonlinear weak shock theory. Three-dimensional effects in both the generation and propagation of the shock waves, however, may be important, over, for example, the full length of the "S" duct on the Lockheed L1011 TriStar.

The paper begins by discussing the mechanism of shock wave detachment from isolated two-dimensional blunt-nosed bodies in supersonic flows. We then derive a simple theory that agrees well with exact calculations. This theory forms the basis of the work on nonuniform fans described in a later section.

We follow this by considering a uniform cascade. A simple analysis, using conservation of mass flow alone, is used to derive a simple relation between the shock wave strength and the blade incidence and thickness. In the next section we look at perturbations of this condition due to blading nonuniformities. The analysis is in two parts: first, the shock wave detachment at the leading edge of the blades is determined and then the propagation of the shock waves forward of the fan is calculated. As a result a simple relation between shock wave strength and the blading nonuniformities is obtained, which is in good agreement with experiments.

II. Detached Shock Waves on Blunt-Nosed Bodies

We consider a semi-infinite body, of thickness t and nose radius r, immersed in a uniform mean flow, of Mach number M. There are a number of ways of estimating the shock detachment distance for such isolated blunt-nosed bodies. First, there is direct calculation using a numerical solution of the flow equations. This is feasible but complicated. Furthermore, it is unlikely to yield either the degree of understanding required here, or some simple relation that could be fitted to an acoustic theory. Additionally, since, as we shall see later, the outflow from the cascade can only be crudely modeled, great accuracy seems unwarranted.

A second method is that of matched expansions. A number of solutions are available (see e.g., Van Dyke[6]). They have two disadvantages. First, they arise from expansions of the flow variables in powers of $1/M_\infty$, and are accordingly excellent for hypersonic speeds ($M_\infty > 4$), but much less suitable for the Mach numbers relevant here ($M_\infty < 1.7$). Second, they mostly solve only the so-called "indirect problems," in which the shape of the shock wave is specified (a hyperbolic shape is frequently assumed) and the body profile then determined.

Our approach is based on the much simpler method due to Möckel,[7] as described in detail in Shapiro.[8] He assumes a shock shape, and finds its position by balancing the mass flows through the sonic line with the upstream mass flow. We adopt this method with a number of simplifications, and obtain a result which agrees well with more exact calculations and, therefore, also (as shown by Shapiro, and by Holder and Chinneck[9]) with experiments for the Mach number range of interest.

We balance the mass flow through the sonic line (see Fig. 2), with that through the characteristic AB. Downstream of the sonic line we assume that the flow is isentropic, and that the characteristics are all of one family. This means that we are in some sense ignoring the rise in entropy through the shock. Since the rise is $0(M-1)^3$ and here $M=1.5$, this is not too important. Since only one characteristic family is present, the position of the sonic line on the body E, may be determined. The sonic line is further assumed to be straight (see Shapiro[8]) and at an angle $(\pi/2-\eta)$ to the mean flow. The mass flow through the characteristic AB is

$$\dot{m} = (\dot{m}\sqrt{T_0}/AP_0)P_0 A_{AB}/T_0 \tag{1}$$

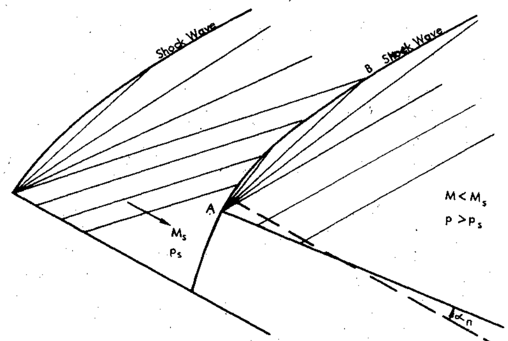

Fig. 1 Effect of blade stagger angle changes; attached shock waves.

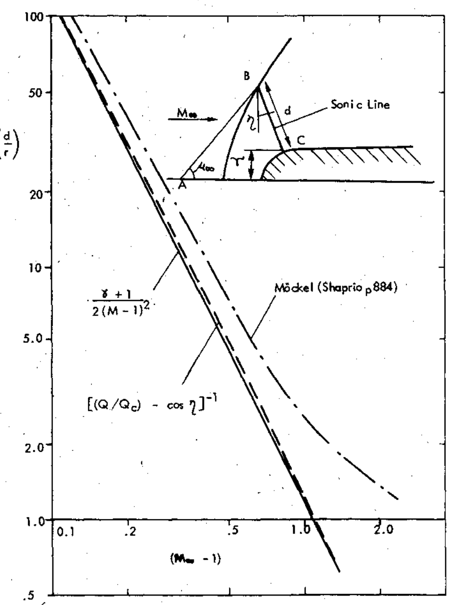

Fig. 2 Shock wave detachment; comparison of theories.

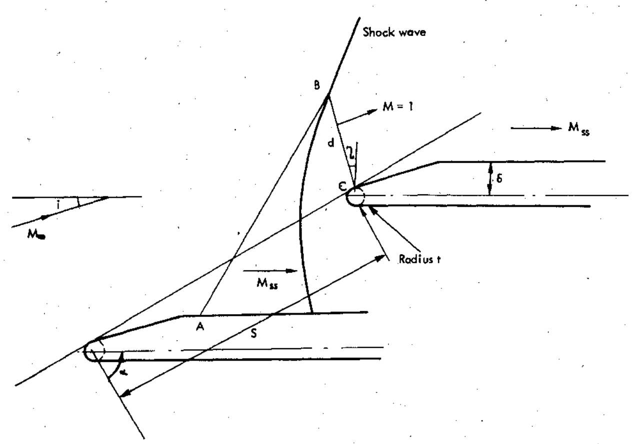

Fig. 3 Notation for calculation of shock wave detachment due to incidence and thickness; uniform blades.

where $(\dot{m}\sqrt{T_0}/AP_0)=Q$ is the one-dimensional mass flow function, P_0, T_0 the stagnation pressure and temperature, and A_{AB} the area of AB perpendicular to the flow (per unit span). This is set equal to the flow through the sonic line,

$$\dot{m}_c = (\dot{m}\sqrt{T_0}/AP_0)P_{0c}A_c/T_0 \tag{2}$$

and we find that with $A_{AB}=r+d\cos\eta$ and $A_c=d$ (per unit span)

$$(r+d\cos\eta)Q_\infty = RQ_c d \tag{3}$$

where R is the average ratio of stagnation pressures across the shock. Hence,

$$d/r = [R(Q_c/Q) - \cos\eta] \tag{4}$$

Figure 3 compares this result with that of Möckel's[7] calculation procedure quoted by Shapiro.[8] We see that the two are in reasonable agreement, except above a Mach number of 2, where the pressure losses become significant. At vey low excess Mach numbers, the shock stand-off distance tends to infinity. This region is probably not well modeled either, since the geometric assumptions are of doubtful validity when the shock is far enough detached. However, the agreement is quite acceptable in the region of real interest.

We examine the form of the result in Eq. (4) for marginally supersonic flow. It can be shown easily that, with $(M-1)=\xi$,

$$Q/Q_c = (1+2\xi^2/(\gamma+1)),\ \cos\eta = 1-0(\xi^2),\ R=0(\xi^3) \tag{5}$$

Hence, the detachment for small excess Mach number is

$$d/r = (\gamma+1)/2(M-1)^2 \tag{6}$$

This is plotted on Fig. 5, and agrees well with Eq. (4) as evaluated exactly.

III. Shock Wave Detachment for a Uniform Cascade

We consider a cascade of blades of stagger angle α and incidence i, leading edge radius t, and thickness δ, at the position where the characteristic AB meets the blade S (Fig. 3). The mass flow through the characteristic AB is then given by

$$\dot{m}_{\mathrm{AB}} = (s\cos\alpha - \delta - (d+t))Q_{ss} \tag{7}$$

In this and subsequent expressions, we have suppressed the constant terms dependent on stagnation conditions. Similarly, the mass flows through the sonic line $\dot{m}_c$ and at infinity $\dot{m}_\infty$ are

$$\dot{m}_c = dR_cQ_c;\ \dot{m} = s\cos(\alpha+i)Q_{ss} \tag{8}$$

Now, the mass flow through AB must be equal to the sum of the mass flow through the sonic line and that through the passage. Since the latter is just the mass flow per passage at infinity, $\dot{m}_{\mathrm{AB}} = \dot{m}_c + \dot{m}_\infty$, and we find that

$$\frac{d}{s} = \frac{[Q_{ss}\cos - Q_c\cos(\alpha+i)] + [(t/s)\cos\eta - (\delta/s)]Q_{ss}}{R_cQ_c - Q_{ss}\cos\eta} \tag{9}$$

There are two parts to this expression dependent on incidence and thickness, respectively. Since the incidence is always small in practice, we can expand the formula for small i. Then changes in Mach number are small and Q_{ss} may be expanded about Q_∞. Using the relations between Mach number and angle for linearized supersonic flow[8] and the influence coefficients relating mass flow and area for one-dimensional flow,[8] it follows that to first order in i,

$$Q_{ss} = Q_\infty - i\beta_\infty Q_\infty \tag{10}$$

where $\beta = \sqrt{M^2-1}$. Hence, to first order in i

$$d/s = i(\sin\alpha - \beta_\infty\cos\alpha)/((R_cQ_c/Q) - \cos\eta) \tag{11}$$

This equation shows that shock detachment measured along the sonic line increases almost linearly with incidence, and is zero for zero incidence, which here is the "unique incidence condition." It may be compared with the more complex analysis of cascade entrance conditions with detached shock waves of York and Woodard.[10]

For the RB211 at 92% speed, the condition where buzz-saw noise is important, we have $M_\infty \doteq 1.4$, $\alpha = 60$ deg, $i=4$ deg, $\eta = 13.4$ deg, and $M_{ss}=1.55$. To estimate R_c, we note that across a normal shock $R_c = 0.913$ at $M_\infty = 1.55$. This must apply to the air which is just spilled over the top of the airfoil. For an oblique shock wave with a downstream Mach number of 1.4, the pressure ratio is $R_c = 0.964$. Accordingly, we estimate the average value of R_c as 0.964. Then using Eq. (9), we find that $d/s = 0.142$.

This is similar to the value calculated by Stratford and Newby. Furthermore, it corresponds to a value of shock pressure rise $(\Delta P/P_\infty) = 0.35$ (actual value 0.42), at half a blade spacing ahead of the fan. This is acceptably close to the measured value of 0.4 in view of both the approximation in the theory and the difficulties of making measurements of this pressure rise, when the flow is unsteady, and there are large blade to blade variations.

The previously given Eq. (9) for shock detachment also shows that if the tip alone is increased in thickness without altering the downstream thickness δ, the change in shock detachment (d/s) is then given by

$$\Delta(d/s) = \cos\eta\Delta(t/s)/[(R_cQ_c/Q_{ss}) - \cos\eta] \tag{12}$$

This is essentially the same formula as that for an isolated airfoil. An interesting feature of Eq. (9), is that if both the tip and upstream thickness increase together (for instance, if there is a uniform thickening of the whole blade), then there is virtually no change in shock detachment.

IV. Shock Detachment for Nonuniform Blading

Effect of Blade Stagger Angle Changes

We consider the geometry shown in Fig. 4, where the blades have perturbations α_n and α_{n-1}, respectively, in stagger angle. We assume for simplicity that the perturbations occur about the same point in each blade. There is no real basis for making this assumption except that if the blades are twisted by different amounts during manufacture, the twist will be about the same axis for each blade.

The basis of the solution is to consider the mass flows through a box formed by the characteristic AB, the sonic line, and the outflow from the blade passage (Fig. 3). These mass flows are balanced by continuity, and are then expanded about their mean values to first order in the perturbation quantities α, Δd. The shock detachment distance can then be determined.

We consider first the change in mass flow through the exit of the cascade. This is assumed to depend only on the area. The reasons for making this assumption are as follows. Physically, since the cascade discharges into what is effectively a constant pressure sink, the outlet pressure on DE must be constant, and unaffected by the nonuniformity, and

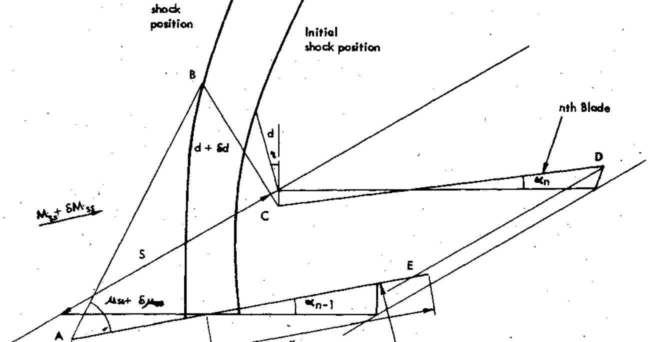

Fig. 4 Notation for calculation of shock wave detachment due to stagger angle variations.

since the flow is nearly sonic the mean flow will be relatively insensitive to any variations in Mach number. As the stagnation temperature and pressure are constant, and dQ/dM is $0((1-M)^2)$, as $M \rightarrow 1$, the mass flow will be proportional to area.

We have justified this further, by solving a linearized problem involving a semi-infinite cascade with the blades moved from their mean position. This is described in detail in Ref. 11 but is too long to reproduce here.

In solving a linear problem it may be objected that the flow is inherently nonlinear at nearly sonic Mach numbers. We maintain, nonetheless, that the linear solutions will be adequate so long as the flow is not singular at $M=1$. The analysis proceeds by the Wiener-Hopf technique.[12] We start by solving the problem for a perfectly general set of perturbations, and generate a set of simultaneous equations, of number equal to the number of blades. We then specialize the analysis to harmonic variations in the stagger angles and displacements. These harmonics essentially are responsible for the harmonics of shaft rotational frequency in the resulting engine order tones. An essential feature of this analysis is that we initially solve for time harmonic variations, and then let the frequency tend to zero. In that respect, the analysis might have some relevance to the flutter problem, and certainly has some similarities to the supersonic flutter work of Goldstein et al.[13] It is also related to much published work on the transmission of sound through blade rows (see, e.g., Mani and Horvay[14]).

This linearized analysis[11] has shown that this is, indeed, asymptotically true (as $M \rightarrow 1$) for low harmonic order variations in the blade positions. The area to be used in these calculations is an effective area (rather than the true, geometric, area), and should account for the presence of the blade boundary layer. We will ignore the variation in the latter with blading nonuniformity.

If these arguments are accepted, we have, in the usual notation, the increase in area per unit span as $(c-x)(\alpha_n - \alpha_{n-1})$. Hence, the change in mass flow is

$$\Delta \dot{m}_{\mathrm{ED}} = (c-x)(\alpha_n - \alpha_{n-1}) R_p A_p Q_p \tag{13}$$

where R_p accounts for the loss in total pressure between the inlet and exit of the box, A_p accounts for the contraction in stream tube height that occurs on a real fan, and represents, therefore, some attempt to take three-dimensional effects into account.

The change in mass flow through the sonic line is

$$\Delta \dot{m}_{\mathrm{BC}} = \Delta d Q_c R_c \tag{14}$$

Hence, there is no stream tube contraction, and, furthermore, we assume that the change in the pressure loss is negligible. This is reasonable, since the loss is in any event a small quantity.

We now consider the mass flow change through AB, which is

$$\begin{aligned}\Delta \dot{m}_{\mathrm{AB}} = {} & [Q_{ss} + \Delta Q_{ss}][(d+\Delta d)\cos(\eta - \alpha_n) \\ & + ((x(\alpha_{n-1} - \alpha_n)/\cos\alpha) + s)\cos(\alpha + \alpha_{n-1})] \\ & - Q_{ss}(d\cos\eta + s\cos\alpha)\end{aligned} \tag{15}$$

Expanding in the perturbation quantities, ΔQ, α_n, α_{n-1}, Δd, we obtain

$$\begin{aligned}\Delta \dot{m}_{\mathrm{AB}} = {} & \Delta Q_{ss}(d\cos\eta + s\cos\alpha) - Q_{ss}[\alpha_{n-1} s\sin\alpha \\ & - \Delta d\cos\eta - \alpha_n d\sin\eta - x(\alpha_{n-1} - \alpha_n)]\end{aligned} \tag{16}$$

We now use the results of Shapiro[8] to express Q in terms of α_{n-1}, as described prior to Eq. (10).

$$(\Delta Q_{ss}/Q_{ss}) = \alpha_{n-1}\beta_{ss} \tag{17}$$

where $\beta_{ss} = \sqrt{M_{ss}^2 - 1}$, and find that

$$\begin{aligned}\Delta \dot{m}_{\mathrm{AB}} = {} & Q_{ss}(\alpha_{n-1}\beta_{ss}(d\cos\eta + s\cos\alpha) - \alpha_{n-1} s\sin\alpha \\ & + \Delta d\cos\eta + x(\alpha_{n-1} - \alpha_n) + \alpha_n d\sin\eta)\end{aligned} \tag{18}$$

Since the net mass inflow into ABCDE must vanish, $\Delta \dot{m}_{\mathrm{AB}} = \Delta \dot{m}_{\mathrm{BC}} + \Delta \dot{m}_{\mathrm{DE}}$, so that, finally,

$$\begin{aligned}\frac{\Delta d}{d} = {} & \Big[\Big(\frac{Q_p c}{Q_c s} R_p A_p + \frac{x}{s}(1 - A_p R_p) - \frac{d}{s}\sin\alpha\Big) \\ & \times (\alpha_{n-1} - \alpha_n) + \Big(\beta_{ss}\Big(\frac{d}{s}\cos\eta + \cos\alpha\Big) + \frac{d}{s}\sin\alpha\Big)\alpha_{n-1}\Big] \\ & \times \Big[\frac{R_c Q_c}{Q_{ss}} - \cos\eta\Big]^{-1}\end{aligned} \tag{19}$$

There are several noteworthy features to this equation. First, it has the multiplying factor $[(R_c Q_c/Q_{ss}) - \cos\eta]^{-1}$ found in all these problems. Second, considering the terms in the numerator, $(1 - R_p A_p)x/s$ is much less than $(c/s)(R_p A_p)$ and may be neglected. Third, $(d/s)\alpha_n \sin\eta$ is also small [both (d/s) and $\sin\eta$ are small] and is neglected.

For the RB211, at the conditions of interest, we estimate the following quantities.

$$R_p A_p = 0.9,\ x/s = 0.6,\ d/s = 0.15,\ c/s = 1.25$$

$$R_c = 0.95,\ \eta = 13\ \mathrm{deg},\ Q_c/Q_{ss} = 1.21,\ M_{ss} = 1.55$$

$$\cos\eta = 0.98,\ \alpha = 60\ \mathrm{deg},\ \beta_{ss} = 1.18$$

The typical value of $\alpha_n = 0.2$ deg. Then, with α_n in degrees

$$\Delta d/d = -0.84(\alpha_n - \alpha_{n-1}) - 0.07\alpha_{n-1} \tag{20}$$

Change in Shock Detachment Due to Blade Thickness Changes

In this section, we discuss the change in shock detachment due to the change in blade thickness. We assume that the cascade of blades is uniform and, therefore, that thickness and stagger are uncoupled. This is a reasonable assumption and likely to be valid, at least for the small changes encountered in practice. The method of analysis is essentially that used earlier. We balance the mass flow through a box bounded by the blades, sonic line, trailing edge plane, and the characteristic from the blade to the sonic line; see Fig. 5.

The change in mass flow through AB is again dependent on area change, which is $-(\delta_n + \delta_{n-1})$. The change in mass flow is simply

$$\Delta \dot{m}_{\mathrm{AB}} = -Q_p A_p R_p(\delta_n - \delta_{n+1}) \tag{21}$$

The result in this form is open to two objections. First, the thicknesses ought really to incorporate the boundary-layer thicknesses, which may not be negligible for these transonic flows. Second, because the streamlines on either side of the blade converge as the flow leaves the blade, the area of the flow should account for the wake and its downstream mixing. Nevertheless, we believe that the assumptions leading to Eq. (21) are sufficiently good for our purposes.

The change in mass flow through the sonic line is

$$\Delta \dot{m}_{\mathrm{BC}} = R_c Q_c \Delta d \tag{22}$$

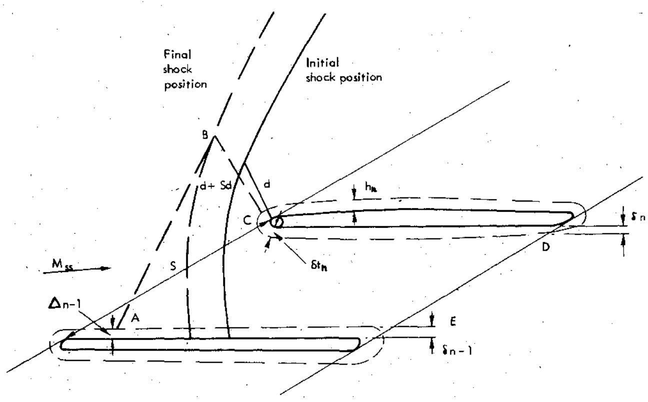

Fig. 5 Notation for calculation of shock wave detachment due to thickness variations; nonuniform blades.

and the change in mass flow through AB is

$$\Delta \dot{m}_{AB} = (-h_{n+1} + (\Delta d_n + \Delta t_n)\cos\eta) Q_{ss} \tag{23}$$

Therefore, since $\Delta \dot{m}_{AB} = \Delta \dot{m}_{BC} + \Delta \dot{m}_{ED}$,

$$d = \frac{[\Delta t_n \cos + (Q_p A_p R_p / Q_{ss})(\delta_n - \delta_{n-1}) - h_{n-1}]}{(Q_c R_c / Q_{ss}) - \cos\eta} \tag{24}$$

This has the usual denominator $[(Q_c R_c / Q_s) - \cos\eta]$, and otherwise depends on three thicknesses: the actual leading-edge thickness, Δt_n, the trailing-edge thickness, δ_n, and the upstream thickness of the preceding blade, h_{n-1}. Because of this dependence on three variables, it is difficult to apply the relation to a fan. Of these three, only the leading-edge thickness is important, since its percentage variation is the largest, the blade leading edges being very thin and hand-finished. Then,

$$\frac{\Delta d}{d} = \frac{(\Delta t_n / d)\cos\eta}{(Q_c R_c / Q_{ss}) - \cos\eta} \tag{25}$$

In this relation, we note from Sec. III that substitution for d gives

$$\frac{\Delta d}{d} = \frac{(\Delta t_n / d)\cos\eta}{\cos\alpha - \cos(\alpha + i) Q_\infty / Q_{ss}} \tag{26}$$

This equation applies when the uniform component of shock standoff distance is dominated by spillage rather than thickness. Using the same values for the parameters as in Sec. III, we find that $\Delta d/d = 33.5(\Delta t_n / s)$. For typical blades, $s = 8$ in. at the tip, and, typically, $\Delta t_n = 0.01$ in. (standard deviation of measurements). This corresponds to $(\Delta d/d) = 0.04$, which is a much smaller percentage change than that due to blade stagger angle variations. We note in passing that it has been found that it is the leading-edge thickness which correlates well with shock strength.

V. Variation of Shock Strength with Shock Detachment

The objective of this section is to determine the variation in the positions of the shock waves as the detachment distance is altered. To do this we consider a shock wave from a single blade, as shown in Fig. 6. In this figure, the shock wave propagates forward into a uniform flow whose characteristics are at a constant angle $\bar{\mu}_s$ to the downstream flow. The coordinates of a point on the shock are r, ψ.

We analyze the problem using weak shock theory.[15] In that theory, the shock is shown to bisect the characteristics in-

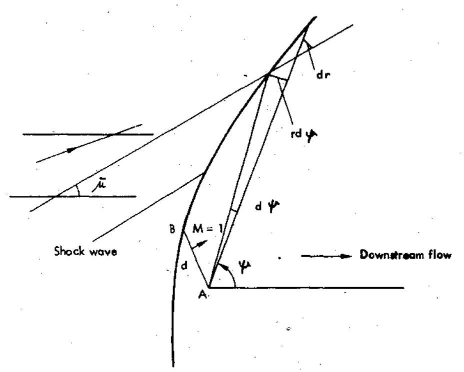

Fig. 6 Notation for shock wave calculation.

tersecting it from upstream and downstream. For the Mach numbers encountered here, this approximation should be adequate since at no point (except perhaps close to the sonic line) are the shock waves in any sense strong. Then, from Fig. 6,

$$\tan\left(\frac{\pi}{2} - \psi + \left(\frac{\psi + \bar{\mu}_s}{2}\right)\right) = -\frac{1}{r}\frac{dr}{d\psi} \tag{27}$$

$$\cot\left(\frac{\psi - \bar{\mu}_s}{2}\right) = -\frac{1}{r}\frac{dr}{d\psi} \tag{28}$$

and integrating from $\psi = \psi_0$, the sonic line, to ψ and from $r = d$ to r, gives

$$\frac{r}{d} = \left[\frac{\sin((\psi_0 - \bar{\mu}_s)/2)}{\sin((\psi - \bar{\mu}_s)/2)}\right]^2 \tag{29}$$

Using this relation, it is relatively straightforward to calculate the change in shock positions with μ_s (which depends on the upstream Mach number), ψ_0 (which depends on the downstream Mach number), and d.

Examining Eq. (29) we find that for the fan under discussion, the maximum value of $(\psi - \bar{\mu}_s)$ occurs at the sonic line, where $\psi = 103$ deg, $\bar{\mu}_s = 40$ deg ($M_s = 1.55$). Then $|(\psi - \bar{\mu}_s)/2| < 32$ deg and we can approximate Eq. (29) as

$$r/d = (\psi_0 - \bar{\mu}_s)/(\psi - \bar{\mu}_s) \tag{30}$$

This equation may be expressed in terms of θ, the angle between the characteristic and the sonic line. Since the sum of this angle and ψ is the same for all Mach numbers and at the upstream Mach number $\psi_s = \bar{\mu}_s$, it follows that, since by definition $(\bar{\mu}_s + \theta_s) = (\psi + \theta)$ and $\theta_0 = 0$, Eq. (30) becomes

$$r/d = (\theta_s / (\theta_s - \theta))^2 \tag{31}$$

Since θ is a function of Mach number alone, this equation may be used to calculate the Mach number, and, hence, the static pressure rise at any position on the shock.

For a uniformly bladed fan, we choose a coordinate system such that $X = r\sin(\psi + \alpha)$, $Y = -r\cos(\psi + \alpha)$. Then X, Y are along the blade leading-edge line and perpendicular to the cascade, α is the stagger angle, and the shape of the shock wave may be expressed in (X, Y) coordinates. It is clear that we now know the pressure rise as a function of (Y/X), using the angle-pressure relations for a Prandtl-Meyer expansion. From this, we can determine the pressure rise relative to the initial pressure $\Delta P / P_\infty$ as a function of Y/d (Fig. 7). This shows the expected decay of the shock strength ahead of the fan disk.

To relate the pressure rise at the shock to the changes in the detachment distance d, we use a small perturbation analysis. To analyze the problem properly we have to account for the

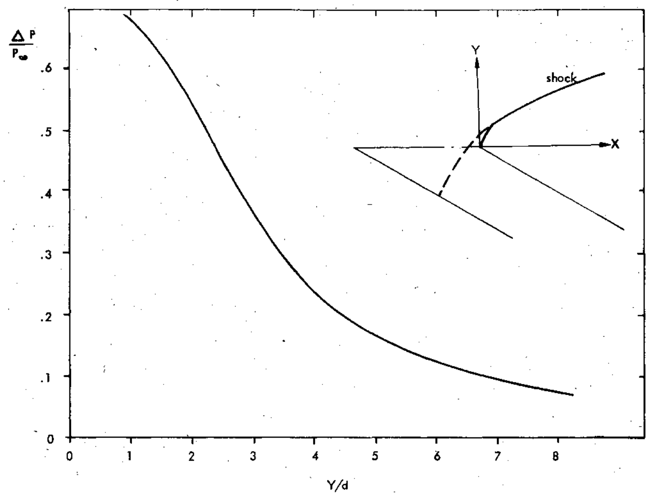

Fig. 7 Decay of shock strength ahead of fan.

changes in both blade orientation and upstream Mach number. However, it can be shown that the effects just referred to are small compared with that of the change in the shock detachment distance. If, therefore, we only account for the change in shock detachment distance d, we can write $\Delta P/P_\infty = f(Y/d)$ and then for small changes we can write

$$\Delta P/P_\infty = -f'(Y/d)(Y/d)(\Delta d/d) \qquad (32)$$

For the fan here, $Y/s=0.5$, $d/s=0.125$, giving $Y/d=4$ and $f'=0.1$. Then $\Delta P/P_\infty = 0.4\,(\Delta d/d)$, and substituting for $\Delta d/d$ from Eq. (20) we obtain the required relation between pressure rise and blade incidences,

$$\Delta P/P_\infty = -0.336(\alpha_n - \alpha_{n-1}) - 0.028\alpha_{n-1} \qquad (33)$$

This is to be compared with the relation determined experimentally by Stratford and Newby.[2]

$$\Delta P/P_\infty = -0.194(\alpha_n - \alpha_{n-1}) \qquad (34)$$

Two things are clear about these results. First, in our relation, there is an extra weak dependence on α_{n-1}, for fixed $(\alpha_n - \alpha_{n-1})$. Second, we have greatly overestimated the rate of change of $(\Delta P/P_\infty)_n$ with $(\alpha_n - \alpha_{n-1})$. There are a number of possible reasons for this. The prime one is the extreme sensitivity of the result to the steady detachment distance. Substituting in Eq. (32) for Δd, and d, we find that

$$\Delta\left(\frac{\Delta P}{P}\right)_n = -f''\left(\frac{Y}{d}\right)\left(\frac{Y}{s}\right)[((Q_cR_c/Q_{ss})-1)((Q_p/Q_{ss}) \times (c/s)R_pA_p + (x/s)(1-A_pR_p))(\alpha_n - \alpha_{n-1})] \div [\cos\alpha - (Q_c/Q_{ss})\cos(\alpha+i)] \qquad (35)$$

In Eq. (35) we have neglected a part proportional to α_{n-1} (in the expression for $\Delta d/d$) and in the numerator, only the $[(Q_p/Q_{ss})(c/s)R_pA_p]$ term is significant, while f' is relatively insensitive to the actual value of d chosen. Now, we note that the numerator and denominator in this expression are both small differences between larger quantities, and the result is, therefore, very sensitive to small changes in the parameters used. In particular, we note the sensitivity to R_c. If R_c were reduced to 0.9, $[(Q_cR_c/Q_{ss}) - \cos\eta]$ would be reduced from 0.17 to 0.112 and the result [Eq. (33)] would be more similar to Stratford and Newby's experimental relation. Also noteworthy is the approximate variation as i^{-2}, indicating sensitivity to this quantity also.

VI. Conclusions

In this paper, we have devised a relatively simple theory for the strengths of the shock waves found ahead of a transonic compressor having nonuniform blading. The theory shows that the shock amplitudes are proportional to the differences between successive blade stagger angles, in agreement with the experimental results of Stratford and Newby. These shock strengths are also dependent on the changes in the thickness of the blades at a number of different points on the blades. Of these thicknesses, that at the leading edge is probably the most important. For typical variations in each, the effect of stagger angle variation is four times that of thickness variation.

While the general dependence of shock amplitude on stagger angle is correctly predicted, the rate of change is not. This is possibly due to the use of inadequate numerical data rather than any defect in the theory itself. The slope of the shock amplitude/stagger angle curve depends on factors that are very sensitive to the conditions used. In particular, it is sensitive to mean blade incidence and to the losses assumed. Neither of these is known accurately in the present context. The strong dependence on incidence does, however, suggest a method of controlling the source of buzz-saw noise. As incidence increases, so does the average shock detachment, but this causes the extra detachment due to the blade nonuniformity to decrease as a percentage of its mean value, with a resulting decrease in the shock amplitude.

There are several ways in which the analysis could be improved. First, it is clear that the correct values of incidence and loss factors are critical, and some way must be found of accurately determining them. Second, and despite the analysis of Ref. 10, one of the most questionable assumptions in the theory is that for the outflow from the cascade. While Ref. 10 justified the assumption that it depended on area alone, it did so on the basis of a linearized analysis, which may be somewhat in error at these high Mach numbers. However, this is likely to be a smaller effect than that of the boundary layer. As the shock strength and position change, so will the boundary-layer thickness and this, in turn, will alter the effective outlet area. To calculate this effect properly would be most difficult. In any event, the analysis here does provide a description of the flow that is consistent with the observed relationship between the shock strengths and the blading nonuniformity. As such, it is about as far as it is worth going with purely analytical means.

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